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AUTHOR(S):

YOSHIZAWA, TARO

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Oscillatory Property for Second Order Differential Equations

Taro Yoshizawa

There are many results on oscillatory property of solutions of differential equations. In this article, we shall discuss oscillatory property of solutions and the existence of a bounded nonoscillatory solution of second order differential equations by applying Liapunov second method.

Consider an equation

$$(1) \quad (r(t)x')' + f(t, x, x') = 0 \quad (' = \frac{d}{dt}),$$

where $r(t) > 0$ is continuous on $I = [0, \infty)$ and $f(t, x, u)$ is continuous on $I \times \mathbb{R} \times \mathbb{R}$, $\mathbb{R} = (-\infty, \infty)$. To discuss oscillatory property of solutions of (1), we consider an equivalent system

$$(2) \quad x' = \frac{y}{r(t)}, \quad y' = -f(t, x, \frac{y}{r(t)}).$$

A solution $x(t)$ of (1) which exists in the future is said to be oscillatory if for every $T > 0$ there is a $t_0 > T$ such that $x(t_0) = 0$. Moreover, the equation (1) is said to be oscillatory if every solution of (1) which exists in the future is oscillatory.

Theorem 1. Assume that there exist two continuous scalar function $V(t, x, y)$ and $W(t, x, y)$ defined on $t \geq T$, $0 < x < K$, $|y| < \infty$ and on $t \geq T$, $-K < x < 0$, $|y| < \infty$, respectively, where T can be large and $K > 0$ or $K = \infty$, and assume that $V(t, x, y)$ and $W(t, x, y)$ satisfy the following conditions;

- (i) $V(t, x, y) \rightarrow \infty$ uniformly for $0 < x < K$ and $-\infty < y < \infty$ as $t \rightarrow \infty$, and
- $W(t, x, y) \rightarrow \infty$ uniformly for $-K < x < 0$ and $-\infty < y < \infty$ as $t \rightarrow \infty$,

(ii) $\dot{V}_{(2)}(t, x(t), y(t)) \leq 0$ for all sufficiently large t , where $\{x(t), y(t)\}$ is a solution of (2) such that $0 < x(t) < K$ for all large t and

$$\dot{V}_{(2)}(t, x(t), y(t)) = \overline{\lim}_{h \rightarrow 0^+} \frac{1}{h} \{ V(t+h, x(t+h), y(t+h)) - V(t, x(t), y(t)) \},$$

(iii) $\dot{W}_{(2)}(t, x(t), y(t)) \leq 0$ for all sufficiently large t , where $\{x(t), y(t)\}$ is a solution of (2) such that $-K < x(t) < 0$ for all large t and

$$\dot{W}_{(2)}(t, x(t), y(t)) = \overline{\lim}_{h \rightarrow 0^+} \frac{1}{h} \{ W(t+h, x(t+h), y(t+h)) - W(t, x(t), y(t)) \}.$$

Then the solution $x(t)$ of (1) such that $|x(t)| < K$ for all large t is oscillatory. Moreover, if $K = \infty$, the equation (1) is oscillatory.

Proof. Let $x(t)$ be a solution of (1) which is defined on $[t_0, \infty)$ and bounded by K for all large t , and suppose that $x(t)$ is not oscillatory. Then $x(t)$ is either positive or negative for all large t . Now assume that $0 < x(t) < K$ for all $t \geq \sigma$ where $\sigma \geq T$. By the condition (i), if t is sufficiently large, say $t \geq t_1$, we have

$$V(\sigma, x(\sigma), y(\sigma)) < V(t, x, y)$$

for all $0 < x < K$, $|y| < \infty$. However, by the condition (ii), we have

$$V(t, x(t), y(t)) \leq V(\sigma, x(\sigma), y(\sigma))$$

for all $t \geq \sigma$, if necessary, choosing a large σ . This contradicts $V(t_1, x(t_1), y(t_1)) > V(\sigma, x(\sigma), y(\sigma))$. When we assume that $-K < x(t) < 0$ for all large t , we have also a contradiction by using $W(t, x(t), y(t))$. Thus we see that $x(t)$ is oscillatory.

To apply this theorem, the following lemmas play an important role. In the following, a scalar function $v(t, x, y)$ will be called a Liapunov function for (2), if $v(t, x, y)$ is continuous in (t, x, y) in the domain of definition and is locally Lipschitzian in (x, y) . Moreover, we define $\dot{v}_{(2)}(t, x, y)$ by

$$\dot{v}_{(2)}(t, x, y) = \overline{\lim}_{h \rightarrow 0^+} \frac{1}{h} \{ v(t+h, x+h \frac{y}{r(t)}, y-hf(t, x, \frac{y}{r(t)})) - v(t, x, y) \}.$$

Lemma 1. For $t \geq T^*$, $x > 0$, $|y| < \infty$, where T^* can be large, we assume that there exists a Liapunov function $v(t, x, y)$ which satisfies the following conditions;

(i) $yv(t, x, y) > 0$ for $t \geq T^*$, $x > 0$, $y \neq 0$,

(ii) $\dot{v}_{(2)}(t, x, y) \leq -\lambda(t)$, where $\lambda(t)$ is a continuous function defined on $t \geq T^*$ and

$$\lim_{t \rightarrow \infty} \int_T^t \lambda(s) ds \geq 0 \text{ for all large } T.$$

Moreover, we assume that there is a τ and a $w(t, x, y)$ for all large T such that $\tau \geq T$ and $w(t, x, y)$ is a Liapunov function defined on $t \geq \tau$, $x > 0$, $y < 0$, which satisfies the following conditions;

(iii) $y \leq w(t, x, y)$ and $w(\tau, x, y) \leq b(y)$, where $b(y)$ is continuous, $b(0) = 0$ and

$$b(y) < 0 \quad (y \neq 0),$$

(iv) $\dot{w}_{(2)}(t, x, y) \leq -\rho(t)w(t, x, y)$, where $\rho(t) \geq 0$ is continuous and

$$\int_{\tau}^{\infty} \frac{1}{r(t)} \exp \left\{ - \int_{\tau}^t \rho(s) ds \right\} dt = \infty.$$

Then, if $\{x(t), y(t)\}$ is a solution of (2) such that $x(t) > 0$ for all large t , we have $y(t) \geq 0$ for all large t .

We can obtain a similar lemma for a solution $\{x(t), y(t)\}$ of (2) such that $x(t) < 0$ for all large t . For the proof of Lemma 1 and the details, see [5].

Proposition 1. For the equation (1) we assume that

$$(i) \quad \int_0^{\infty} \frac{dt}{r(t)} = \infty,$$

(ii) for $t \geq 0$ and $x \geq 0$, there exist continuous functions $a(t)$ and $a(x)$ such that

$$(3) \quad \lim_{t \rightarrow \infty} \int_T^t a(s) ds \geq 0 \quad \text{for all large } T$$

and that $xa(x) > 0$ ($x \neq 0$), $a'(x) \geq 0$ and for all large t , $x \geq 0$, $|u| < \infty$

$$a(t)a(x) \leq f(t, x, u),$$

(iii) for $t \geq 0$ and $x \leq 0$, there exist continuous functions $b(t)$ and $\beta(x)$ such that

$$\lim_{t \rightarrow \infty} \int_T^t b(s) ds \geq 0 \quad \text{for all large } T$$

and that $x\beta(x) > 0$ ($x \neq 0$), $\beta'(x) \geq 0$ and for all large t , $x \leq 0$, $|u| < \infty$

$$f(t, x, u) \leq b(t)\beta(x).$$

Then, if $\int_0^{\infty} a(t) dt = \infty$ and $\int_0^{\infty} b(t) dt = \infty$, the equation (1) is oscillatory. Moreover, if we have

$$(4) \quad \int_0^{\infty} a(t) dt < \infty, \quad \int_0^{\infty} \left(\frac{1}{r(s)} \int_s^{\infty} a(u) du \right) ds = \infty$$

and

$$(5) \quad \int_0^{\infty} b(t) dt < \infty, \quad \int_0^{\infty} \left(\frac{1}{r(s)} \int_s^{\infty} b(u) du \right) ds = \infty,$$

then all bounded solutions of (1) are oscillatory. In addition to the conditions above, if

$$(6) \quad \int_{\epsilon}^{\infty} \frac{du}{a(u)} < \infty, \quad \int_{-\epsilon}^{-\infty} \frac{du}{\beta(u)} < \infty \quad \text{for some } \epsilon > 0,$$

the equation (1) is oscillatory.

Proof. Under our assumptions, if we consider a function $v(t, x, y) = \frac{y}{a(x)}$ for large t , this function satisfies the conditions in Lemma 1 with $\lambda(t) = a(t)$. Since the condition (3) implies that for all large T , there is a τ such that $\tau \geq T$ and $\int_{\tau}^t a(s)ds \geq 0$ for all $t \geq \tau$, a function $w(t, x, y) = y + a(x) \int_{\tau}^t a(s)ds$ defined on $t \geq \tau$, $x > 0$, $y < 0$ satisfies the conditions in Lemma 1 with $\rho(t) \equiv 0$. Thus we can see that if $\{x(t), y(t)\}$ is a solution of (2) such that $x(t) > 0$ for all large t , then $y(t) \geq 0$ for all large t . We can also see that for a solution such that $x(t) < 0$ for all large t , $y(t) \leq 0$ for all large t .

In the case where we assume that $\int_0^{\infty} a(t)dt = \infty$ and $\int_0^{\infty} b(t)dt = \infty$, for large t , if we define $V(t, x, y)$ and $W(t, x, y)$ by

$$V(t, x, y) = \begin{cases} \frac{y}{a(x)} + \int_0^t a(s)ds & (y \geq 0) \\ \int_0^t a(s)ds & (y < 0) \end{cases}$$

and

$$W(t, x, y) = \begin{cases} \int_0^t b(s)ds & (y > 0) \\ \frac{y}{\beta(x)} + \int_0^t b(s)ds & (y \leq 0), \end{cases}$$

we can see that these functions satisfy the conditions in Theorem 1 for $K = \infty$, and hence the equation (1) is oscillatory.

In the case where we assume (4) and (5), letting $K > 0$ be a constant, se.

$$V(t, x, y) = \int_x^K \frac{du}{a(u)} + \int_0^t \left(\frac{1}{r(s)} \int_s^\infty a(u) du \right) ds$$

for $t \geq 0$, $0 < x < K$ and $|y| < \infty$. For a solution $x(t)$ of (2) which satisfies $0 < x(t) < K$ for all large t , there is a $\sigma > 0$ such that $0 < x(t) < K$ and $y(t) \geq 0$ for $t \geq \sigma$, and hence

$$\dot{V}_{(2)}(t, x(t), y(t)) = \frac{1}{r(t)} \left\{ -\frac{y(t)}{a(x(t))} + \int_t^\infty a(u) du \right\}.$$

If we set $V^*(t, x, y) = -\frac{y}{a(x)} + \int_t^\infty a(u) du$, we have $\overline{\lim}_{t \rightarrow \infty} V^*(t, x(t), y(t)) \leq 0$. On the other hand, we have $\dot{V}_{(2)}^*(t, x, y) \geq 0$, and hence $V^*(t, x(t), y(t)) \leq 0$, which implies that $\dot{V}_{(2)}(t, x(t), y(t)) \leq 0$ for $t \geq \sigma$. For $t \geq 0$, $-K < x < 0$ and $|y| < \infty$, define $W(t, x, y)$ by

$$W(t, x, y) = \int_x^{-K} \frac{du}{\beta(u)} + \int_0^t \left(\frac{1}{r(s)} \int_s^\infty b(u) du \right) ds.$$

Then the conclusion follows from Theorem 1, because K is arbitrary.

In addition, when we assume (6), we can set $K = \infty$ in $V(t, x, y)$ and $W(t, x, y)$ above, and hence the equation (1) is oscillatory.

The result above contains Coles' result [2] and Macki and Wong's result [3].

Remark. It is clear that we can combine the conditions on $a(t)$ and $b(t)$. The Liapunov's method is also applicable to obtain Bobisud's [1] and Opial's [4] results, see [5].

Now we shall discuss the existence of a bounded nonoscillatory solution of (1). The following theorems will be applied. Consider an equation of the second order

$$(7) \quad x'' = F(t, x, x'),$$

where $F(t, x, y)$ is continuous on $I \times \mathbb{R} \times \mathbb{R}$. Let $\underline{\omega}(t)$ and $\overline{\omega}(t)$ be two functions defined

on I , twice differentiable and bounded on I with their derivatives. We assume that $\underline{\omega}(t) \leq \bar{\omega}(t)$,

$$(8) \quad \bar{\omega}''(t) \leq F(t, \bar{\omega}(t), \bar{\omega}'(t))$$

and

$$(9) \quad \underline{\omega}''(t) \geq F(t, \underline{\omega}(t), \underline{\omega}'(t))$$

for all $t \geq 0$.

Theorem 2. Suppose that there exist two Liapunov functions $V(t, x, y)$ and $W(t, x, y)$ defined on $0 \leq t < \infty$, $\underline{\omega}(t) \leq x \leq \bar{\omega}(t)$, $y \geq K$ and on $0 \leq t < \infty$, $\underline{\omega}(t) \leq x \leq \bar{\omega}(t)$, $y \leq -K$, respectively, where $K > 0$ can be large, and assume that $V(t, x, y)$ and $W(t, x, y)$ satisfy the following conditions;

- (i) $V(t, x, y) \leq b(y)$ and $W(t, x, y) \leq b(|y|)$, where $b(r) > 0$ is continuous,
- (ii) $V(t, x, y) \rightarrow \infty$ as $y \rightarrow \infty$, $W(t, x, y) \rightarrow \infty$ as $y \rightarrow -\infty$, uniformly for t, x ,
- (iii) in the interior of their domains of definition

$$\dot{V}(t, x, y) = \overline{\lim}_{h \rightarrow 0^+} \frac{1}{h} \{ V(t+h, x+hy, y+hF(t, x, y)) - V(t, x, y) \} \geq 0$$

and

$$\dot{W}(t, x, y) = \overline{\lim}_{h \rightarrow 0^+} \frac{1}{h} \{ W(t+h, x+hy, y+hF(t, x, y)) - W(t, x, y) \} \leq 0$$

or

- (iii)' in the interior of their domains of definition

$$\dot{V}(t, x, y) \leq 0 \quad \text{and} \quad \dot{W}(t, x, y) \geq 0.$$

Then the equation (7) has a solution $x(t)$ such that $\underline{\omega}(t) \leq x(t) \leq \bar{\omega}(t)$ and $x'(t)$ is bounded for all $t \geq 0$.

Theorem 3. Under the assumptions in Theorem 2, if $\underline{\omega}(0) = \overline{\omega}(0)$ and

$$\dot{V}(t, x, y) \leq 0 \quad \text{and} \quad \dot{W}(t, x, y) \leq 0$$

in the interiors of their domains of definition, then the equation (7) has a solution $x(t)$ such that $\underline{\omega}(t) \leq x \leq \overline{\omega}(t)$ and $x'(t)$ is bounded for all $t \geq 0$.

For the proofs, see [6].

In discussing the existence of a bounded nonoscillatory solution of (1), we assume that the derivative of $r(t)$ is continuous, and consequently the equation (1) can be written as

$$(10) \quad x'' + \frac{r'(t)}{r(t)} x' + \frac{1}{r(t)} f(t, x, x') = 0.$$

Proposition 2. Suppose that there exist functions $b(t)$ and $\beta(x)$ which satisfy the following conditions;

- (i) $b(t)$ is continuous on I and $b(t) \geq 0$ for $t \geq T$, where T can be large,
- (ii) $\beta(x)$ is continuous on $x \geq 0$,
- (iii) for $t \geq T$, $x > 0$ and all y ,

$$(11) \quad f(t, x, y) \leq b(t)\beta(x).$$

Moreover, we assume that there is a $c > 0$ such that $f(t, c, 0) \geq 0$ for $t \geq T$. Then, if

$$(12) \quad 0 < \epsilon \leq r(t) \leq \rho \quad \text{for some } \epsilon, \rho \text{ and all } t \geq 0$$

and

$$(13) \quad \int_0^\infty tb(t)dt < \infty$$

or if, there is an $A > 0$ such that

$$(14) \quad \left| \frac{r'(t)}{r(t)} \right| < A \quad \text{for } t \geq 0$$

and we have

$$(15) \quad \int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty b(u) du \right) ds < \infty,$$

the equation (10) has a bounded nonoscillatory solution.

Proof. Under the conditions (12) and (13), the condition (13) implies that $\int_0^\infty b(t) dt < \infty$, and consequently $\int_s^\infty b(u) du$ exists and is small if s is sufficiently large, because $b(t) \geq 0$ eventually. Since $\epsilon \leq r(t) \leq \rho$, we have

$$\int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty b(u) du \right) ds < \infty \quad \text{and} \quad \frac{1}{r(t)} \int_t^\infty b(u) du < \infty.$$

There is an $L > 0$ such that $\beta(c) \leq \frac{L}{2}$ and there is a $\delta > 0$ such that $\beta(x) \leq L$ if $|x - c| \leq \delta$.

Choose $t_0 \geq T$ so large that

$$0 \leq L \int_{t_0}^t \left(\frac{1}{r(s)} \int_s^\infty b(u) du \right) ds \leq \delta \quad \text{for all } t \geq t_0.$$

For $t_0 \leq t < \infty$, define $\underline{\omega}(t)$ and $\overline{\omega}(t)$ by

$$\underline{\omega}(t) = c \quad \text{and} \quad \overline{\omega}(t) = c + L \int_{t_0}^t \left(\frac{1}{r(s)} \int_s^\infty b(u) du \right) ds.$$

Then $0 < \underline{\omega}(t) \leq \overline{\omega}(t) \leq c + \delta$ for all $t \geq t_0$, and $\underline{\omega}(t)$, $\overline{\omega}(t)$ are bounded with their derivatives.

Clearly we have $\underline{\omega}''(t) \geq -\frac{r'(t)}{r(t)} \underline{\omega}'(t) - \frac{1}{r(t)} f(t, \underline{\omega}(t), \underline{\omega}'(t))$. On the other hand, $\overline{\omega}'(t) = \frac{L}{r(t)} \int_t^\infty b(u) du$ and $\overline{\omega}''(t) = -\frac{r'(t)}{r^2(t)} L \int_t^\infty b(u) du - \frac{L}{r(t)} b(t)$. Thus, using (11),

we have

$$\begin{aligned}
 -\frac{r'(t)}{r(t)} \bar{\omega}'(t) - \frac{1}{r(t)} f(t, \bar{\omega}(t), \bar{\omega}'(t)) &\geq -\frac{r'(t)}{r^2(t)} L \int_t^\infty b(u) du - \frac{1}{r(t)} b(t) \beta(\bar{\omega}(t)) \\
 &\geq -\frac{r'(t)}{r^2(t)} L \int_t^\infty b(u) du - \frac{L}{r(t)} b(t) \\
 &\geq \bar{\omega}''(t),
 \end{aligned}$$

since $c \leq \bar{\omega}(t) \leq c + \delta$ for all $t \geq t_0$ and hence $\beta(\bar{\omega}(t)) \leq L$ for $t \geq t_0$.

For $t \geq t_0$, $\underline{\omega}(t) \leq x \leq \bar{\omega}(t)$ and $y \geq K$, define $V(t, x, y)$ by

$$V(t, x, y) = L \int_{t_0}^t b(s) ds + r(t)y$$

and for $t \geq t_0$, $\underline{\omega}(t) \leq x \leq \bar{\omega}(t)$ and $y \leq -K$, define $W(t, x, y)$ by

$$W(t, x, y) = -L \int_{t_0}^t b(s) ds - r(t)y.$$

Then it is clear that $V(t, x, y)$ and $W(t, x, y)$ satisfy the conditions (i) and (ii) in Theorem 2.

Since $\beta(x) \leq L$ for $\underline{\omega}(t) \leq x \leq \bar{\omega}(t)$, we have

$$\begin{aligned}
 \dot{V}(t, x, y) &= Lb(t) + r'(t)y + r(t) \left\{ -\frac{r'(t)}{r(t)} y - \frac{1}{r(t)} f(t, x, y) \right\} \\
 &= Lb(t) - f(t, x, y) \\
 &\geq Lb(t) - b(t)\beta(x) \geq Lb(t) - Lb(t) = 0
 \end{aligned}$$

and we have also $\dot{W}(t, x, y) \leq 0$. Therefore, it follows from Theorem 2 that the equation (10) has a solution $x(t)$ such that

$$0 < c \leq x(t) \leq c + \delta \quad \text{for all } t \geq t_0$$

and that $x'(t)$ is bounded for all $t \geq t_0$.

Under the conditions (14) and (15), we can use the same $\underline{\omega}(t)$ and $\overline{\omega}(t)$, since (14) and (15) imply that $\frac{1}{r(t)} \int_t^\infty b(u) du < \infty$. Moreover, (14) and (15) imply that $\int_0^\infty \frac{b(s)}{r(s)} ds < \infty$, and hence it is sufficient to consider

$$V(t, x, y) = y + Ax + L \int_{t_0}^t \frac{b(s)}{r(s)} ds$$

and

$$W(t, x, y) = -y + Ax - L \int_{t_0}^t \frac{b(s)}{r(s)} ds.$$

Remark. Assuming the existence of functions $a(t)$ and $a(x)$ such that $a(t)a(x) \leq f(t, x, y)$ for $t \geq T$, $x < 0$ and all y , we can obtain a result similar to Proposition 2.

By applying Theorem 3, we shall now prove the following proposition.

Proposition 3. Suppose that there exist two functions $b(t)$ and $\beta(x, y)$ which satisfy the following conditions;

- (i) $b(t)$ is continuous on I and $b(t) \geq 0$ for $t \geq T$, where T can be large,
- (ii) $\beta(x, y)$ is continuous on $x \geq 0$ and $y \geq 0$,
- (iii) for $t \geq T$, $x > 0$ and $y \geq 0$,

$$(16) \quad f(t, x, y) \leq b(t)\beta(x, y).$$

Moreover, we assume that there is a $c > 0$ such that

$$(17) \quad f(t, c, 0) \geq 0 \quad \text{for } t \geq T$$

and that for some $M > 0$ such that $c < M$

$$(18) \quad f(t, x, y) \geq 0 \quad \text{for } t \geq T, \quad M \geq x \geq c \quad \text{and } y > K,$$

$$(19) \quad f(t, x, y) \leq 0 \quad \text{for } t \geq T, \quad M \geq x \geq c \quad \text{and } y < -K,$$

where K can be large. If $0 < \epsilon \leq r(t) \leq \rho$ for some ϵ, ρ and all $t \geq 0$ and

$$(20) \quad \int_0^\infty t b(t) dt < \infty$$

or if there is an $A > 0$ such that

$$(21) \quad -A < \frac{r'(t)}{r(t)} \quad \text{for all } t \geq 0$$

and we have

$$(22) \quad \int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty b(u) du \right) ds < \infty,$$

then the equation (10) has a bounded nonoscillatory solution.

Proof. In both cases, we have (22) and the fact that $\frac{1}{r(t)} \int_t^\infty b(u) du \rightarrow 0$ as $t \rightarrow \infty$. For the c , $\beta(c, 0) \leq \frac{L}{2}$ for some L . Since $\beta(x, y)$ is continuous, there is a $\delta > 0$ such that if $|x - c| \leq \delta$ and $0 \leq y \leq \delta$, we have $\beta(x, y) \leq L$. Choose $t_0 \geq T$ so large that

$$L \int_{t_0}^t \left(\frac{1}{r(s)} \int_s^\infty b(u) du \right) ds \leq \min(M - c, \delta) \quad \text{for } t \geq t_0$$

and

$$L \frac{1}{r(t)} \int_t^\infty b(u) du \leq \delta \quad \text{for } t \geq t_0.$$

Then, in the same way as in the proof of Proposition 2, we can see that

$$\underline{\omega}(t) \equiv c \quad \text{and} \quad \overline{\omega}(t) = c + L \int_{t_0}^t \left(\frac{1}{r(s)} \int_s^\infty b(u) du \right) ds$$

satisfy the conditions (8) and (9) for $t \geq t_0$.

In the case where we have $\epsilon \leq r(t) \leq \rho$, for $t \geq t_0$, $\underline{\omega}(t) \leq x \leq \bar{\omega}(t)$ and $y > K$, define $V(t, x, y)$ by $V(t, x, y) = r(t)y$. Then we have

$$\begin{aligned} \dot{V}(t, x, y) &= r'(t)y + r(t) \left\{ -\frac{r'(t)}{r(t)}y - \frac{1}{r(t)}f(t, x, y) \right\} \\ &= -f(t, x, y) \\ &\leq 0. \end{aligned}$$

For $t \geq t_0$, $\underline{\omega}(t) \leq x \leq \bar{\omega}(t)$ and $y < -K$, $W(t, x, y) = -r(t)y$ satisfies $\dot{W}(t, x, y) \leq 0$.

In the case where we have the condition (21), $V(t, x, y) = y - Ax$ and $W(t, x, y) = -y + Ax$ are Liapunov functions that we desire. Therefore, it follows from Theorem 3 that the equation (10) has a bounded nonoscillatory solution.

Remark. Assuming the existence of functions $a(t)$ and $a(x, y)$ such that $a(t)a(x, y) \leq f(t, x, y)$ for $t \geq T$, $x < 0$ and $y \leq 0$, we can obtain a result similar to Proposition 3.

Now consider the equation (1). We assume that there exist continuous functions $a(t)$, $b(t)$, $a(x)$ and $\beta(x)$ which satisfy the following conditions;

- (i) $a(t)$ and $b(t)$ are nonnegative for $t \geq T$, where T can be large,
- (ii) $xa(x) > 0$ and $x\beta(x) > 0$ for $x \neq 0$, and $a'(x) \geq 0$, $\beta'(x) \geq 0$,
- (iii) $a(t)a(x) \leq f(t, x, u) \leq b(t)\beta(x)$ for $t \geq T$, $|x| < \infty$ and $|u| < \infty$.

Moreover, we assume that the derivative of $r(t)$ is continuous and that $0 < \epsilon \leq r(t) \leq \rho$ for some ϵ, ρ and all $t \geq 0$.

Under the assumptions above, the following results follow immediately from Propositions 1 and 2 with the remark.

Proposition 4. A necessary and sufficient condition in order that all bounded solutions of (1) are oscillatory is that

$$\int_0^{\infty} ta(t)dt = \infty \quad \text{and} \quad \int_0^{\infty} tb(t)dt = \infty.$$

Proposition 5. A necessary and sufficient condition in order that the equation (1) has a bounded nonoscillatory solution is that

$$\int_0^{\infty} ta(t)dt < \infty \quad \text{or} \quad \int_0^{\infty} tb(t)dt < \infty.$$

Proposition 6. In addition to the assumptions above, if

$$\int_{\epsilon}^{\infty} \frac{du}{a(u)} < \infty \quad \text{and} \quad \int_{-\epsilon}^{-\infty} \frac{du}{\beta(u)} < \infty \quad \text{for some } \epsilon > 0,$$

a necessary and sufficient condition in order that the equation (1) is oscillatory is that

$$\int_0^{\infty} ta(t)dt = \infty \quad \text{and} \quad \int_0^{\infty} tb(t)dt = \infty.$$

For the equation (1), we assume that there exist continuous functions $a(t)$, $b(t)$, $a(x)$ and $\beta(x)$ mentioned above. Moreover, we assume that the derivative of $r(t)$ is continuous, $\int_0^{\infty} \frac{dt}{r(t)} = \infty$ and there is an $A > 0$ such that $\left| \frac{r'(t)}{r(t)} \right| < A$ for $t \geq 0$. Then, by Proposition 1 if we have the condition A that

$$(23) \quad \begin{cases} \int_0^{\infty} a(s)ds = \infty \\ \text{or} \\ \int_0^{\infty} a(s)ds < \infty, \quad \int_0^{\infty} \left(\frac{1}{r(s)} \int_s^{\infty} a(u)du \right) ds = \infty \end{cases}$$

and

$$(24) \quad \begin{cases} \int_0^\infty b(s)ds = \infty \\ \text{or} \\ \int_0^\infty b(s)ds < \infty, \quad \int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty b(u)du \right) ds = \infty, \end{cases}$$

all bounded solutions of (1) are oscillatory. On the other hand, if we have the condition B that

$$(25) \quad \int_0^\infty a(s)ds < \infty \quad \text{and} \quad \int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty a(u)du \right) ds < \infty$$

or

$$(26) \quad \int_0^\infty b(s)ds < \infty \quad \text{and} \quad \int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty b(u)du \right) ds < \infty,$$

the equation (1) has a bounded nonoscillatory solution. Therefore the condition A is a necessary and sufficient condition in order that all bounded solutions of (1) are oscillatory, and the condition B is a necessary and sufficient condition in order that the equation (1) has a bounded nonoscillatory solution.

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